

ZASSENHAUS CONJECTURE FOR CENTRAL EXTENSIONS OF S_5

VICTOR BOVDI AND MARTIN HERTWECK

ABSTRACT. We confirm a conjecture of Zassenhaus about rational conjugacy of torsion units in integral group rings for a covering group of the symmetric group S_5 and for the general linear group $GL(2, 5)$.

1. INTRODUCTION

The conjecture of the title states:

(ZC1) For a finite group G , every torsion unit in $\mathbb{Z}G$ is conjugate to an element of $\pm G$ in the units of $\mathbb{Q}G$.

It remains not only unsolved but also lacking in plausible means of either proof or counter-example, at least for non-solvable groups G . The purpose of this note is to add two further groups to the small list of non-solvable groups G for which conjecture (ZC1) has been verified (see [9, 11, 16, 17]).

Example 1. The conjecture (ZC1) holds true for the covering group \tilde{S}_5 of the symmetric group S_5 which contains a unique conjugacy class of involutions.

Example 2. The conjecture (ZC1) holds true for the general linear group $GL(2, 5)$.

We remark that $PGL(2, 5) \cong S_5$ (see [14, Kapitel II, 6.14 Satz]).

The covering group \tilde{S}_5 occurs as Frobenius complement in Frobenius groups (for the classification of Frobenius complements see [20]). From already existing work [8, 9, 15] it follows that Example 1 supplies the missing bit for the proof of:

Theorem 3. *Let G be a finite Frobenius group. Then each torsion unit in $\mathbb{Z}G$ which is of prime power order is conjugate to an element of $\pm G$ in the units of $\mathbb{Q}G$.*

The proofs are obtained by applying the Luthar–Passi method [11, 16]. We dispose of the validity of (ZC1) for S_5 , established in [17] (see also [11, Section 4] for a proof using the Luthar–Passi method). Below, we briefly recall this method, which uses the character table and/or modular character tables in an automated process suited for being done on a computer, the result being that rational conjugacy of torsion units of a given order to group elements is either proven or not, and if not, one gets at least some information about partial augmentations (cf. [3–5]). We tried to prevent the proofs from useless ballast and to present them in human-readable format, rather than producing systems of inequalities and their solutions which should be reasonably done on a computer (cf. [6]).

⁰The research was supported by OTKA T 037202, T 038059

2000 *Mathematics Subject Classification.* Primary 16S34, 16U60; Secondary 20C05.

Key words and phrases. integral group ring, torsion unit, Zassenhaus conjecture.

Let G be a finite group. Recall that for a group ring element $u = \sum_{g \in G} a_g g$ (all $a_g \in \mathbb{Z}$), its partial augmentation with respect to the conjugacy class x^G of an element x of G , in the following denoted by $\varepsilon_x(u)$ or $\varepsilon_{x^G}(u)$, is $\sum_{g \in x^G} a_g$. When dealing with conjecture (ZC1), it suffices to consider units of augmentation one which form a group denoted by $V(\mathbb{Z}G)$.

A few preliminary remarks on a torsion unit u in $V(\mathbb{Z}G)$ seem to be appropriate.

The familiar result of Berman–Higman (from [1] and [13, p. 27]) asserts that if $\varepsilon_z(g) \neq 0$ for some z in the center of G , then $u = z$.

A practical criterion for u being conjugate to an element of G in the units of $\mathbb{Q}G$ is that all but one of the partial augmentations of every power of u must vanish (see [18, Theorem 2.5]). The converse is obvious.

The next remarks will be used repeatedly.

Remark 4. Let u be a torsion unit in $V(\mathbb{Z}G)$, let $N \trianglelefteq G$ and set $\bar{G} = G/N$. We write \bar{u} for the image of u under the natural map $\mathbb{Z}G \rightarrow \mathbb{Z}\bar{G}$. Since any conjugacy class of G maps onto a conjugacy class of \bar{G} , we have that for any $x \in G$, the partial augmentation $\varepsilon_{\bar{x}}(\bar{u})$ is the sum of the partial augmentations $\varepsilon_{x^G}(u)$ with $g \in G$ such that \bar{g} is conjugate to \bar{x} in \bar{G} .

Now suppose that N is a central subgroup of G , and that $\bar{u} = 1$. Then $u \in N$. Indeed, $1 = \varepsilon_1(\bar{u}) = \sum_{n \in N} \varepsilon_n(u)$, so u has a central group element in its support and the Berman–Higman result applies.

Remark 5. Let u be a torsion unit in $V(\mathbb{Z}G)$. Then $g \in G$ and $\varepsilon_g(u) \neq 0$ implies that the order of g divides the order of u . Indeed, it is well known that then prime divisors of the order of g divide the order of u (see [18, Theorem 2.7], as well as [12, Lemma 2.8] for an alternative proof). Further, it was observed in [11, Lemma 5.6] that the orders of the p -parts of g cannot exceed those of u .

We briefly recall the Luthar–Passi method. Let $u \in V(\mathbb{Z}G)$. Suppose that $u^n = 1$ for some natural number n and let ζ be a primitive complex n -th root of unity. Let χ be the character afforded by a complex representation D of G , and write $\mu(\xi, u, \chi)$ for the multiplicity of an n -th root of unity ξ as an eigenvalue of the matrix $D(u)$. Then (cf. [16], [11, Section 3]):

$$\mu(\xi, u, \chi) = \frac{1}{n} \sum_{d|n} \text{Tr}_{\mathbb{Q}(\zeta^d)/\mathbb{Q}}(\chi(u^d) \xi^{-d}).$$

When trying to show that u is rationally conjugate to an element of G , one may assume—by induction on the order of u —that the values of the summands for $d \neq 1$ are “known.” The summand for $d = 1$ can be written as

$$\frac{1}{n} \sum_{g \in G} \varepsilon_g(u) \text{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\chi(g) \xi^{-1}),$$

a linear combination of the $\varepsilon_g(u)$ with “known” coefficients. Note that the $\mu(\xi, u, \chi)$ are non-negative integers, bounded above by $\chi(1)$. Thus in some sense, there are linear inequalities in the partial augmentations of u which impose constraints on them. Trying to make use of these inequalities is now understood as being the Luthar–Passi method.

A modular version of this method (see [11, Section 3] for details) can be derived from the following observation in the very same way as the original (complex) version is derived from the (obvious) fact that $\chi(u) = \sum_{g \in G} \varepsilon_g(u) \chi(g)$. Suppose

that p is a rational prime which does not divide the order of u (i.e., u is a p -regular torsion unit). Then for every Brauer character φ of G (relative to p) we have (see [11, Theorem 2.2]):

$$\varphi(u) = \sum_{\substack{g^G: g \text{ is} \\ p\text{-regular}}} \varepsilon_g(u) \varphi(g).$$

Thereby, the domain of φ is naturally extended to the set of p -regular torsion units in $\mathbb{Z}G$.

2. A COVERING GROUP OF S_5

A presentation of a covering group of S_n is given by

$$\begin{aligned} \tilde{S}_n = \langle g_1, \dots, g_{n-1}, z \mid & g_i^2 = (g_j g_{j+1})^3 = (g_k g_l)^2 = z, z^2 = [z, g_i] = 1 \\ & \text{for } 1 \leq i \leq n-1, 1 \leq j \leq n-2, k \leq l-2 \leq n-3 \rangle. \end{aligned}$$

Recent results in the representation theory of the covering groups of symmetric groups can be found in Bessenrodt's survey article [2]. We merely remark that the complex spin characters of \tilde{S}_n , i.e., those characters which are not characters of S_n , were determined by Schur [21].

The group \tilde{S}_5 has catalogue number 89 in the Small Group Library in GAP [10] (the other covering group of S_5 has number 90). The spin characters of \tilde{S}_5 as produced by GAP are shown in Table 1 (dots indicate zeros).

	1a	5a	4a	2a	10a	6a	3a	8a	8b	4b	12a	12b
χ_5	4	-1	.	-4	1	2	-2
χ_6	4	-1	.	-4	1	-1	1	.	.	.	β	$-\beta$
χ_7	4	-1	.	-4	1	-1	1	.	.	.	$-\beta$	β
χ_{11}	6	1	.	-6	-1	.	.	α	$-\alpha$.	.	.
χ_{12}	6	1	.	-6	-1	.	.	$-\alpha$	α	.	.	.

Irrational entries: $\alpha = -\zeta_8 + \zeta_8^3 = -\sqrt{2}$ where $\zeta_8 = \exp(2\pi i/8)$,
 $\beta = \zeta_{12}^7 - \zeta_{12}^{11} = -\sqrt{3}$ where $\zeta_{12} = \exp(2\pi i/12)$.

TABLE 1. Spin characters of \tilde{S}_5

We turn to the proof that conjecture (ZC1) holds true for \tilde{S}_5 . Let z be the central involution in \tilde{S}_5 . Then we have a natural homomorphism $\pi : \mathbb{Z}\tilde{S}_5 \rightarrow \mathbb{Z}\tilde{S}_5/\langle z \rangle = \mathbb{Z}S_5$. Let u be a nontrivial torsion unit in $V(\mathbb{Z}\tilde{S}_5)$. We shall show that all but one of its partial augmentations vanish. Since (ZC1) is true for S_5 , the order of $\pi(u)$ agrees with the order of an element of S_5 , and it follows that the order of u agrees with the order of an element of \tilde{S}_5 (see Remark 4). By the Berman–Higman result we can assume that $\varepsilon_1(u) = 0$ and $\varepsilon_z(u) = 0$. Further, we can assume that the order of u is even since otherwise rational conjugacy of u to an element of G follows from the validity of (ZC1) for S_5 and [8, Theorem 2.2]. Denote the partial augmentations of u by $\varepsilon_{1a}, \varepsilon_{5a}, \dots, \varepsilon_{12b}$ (so that ε_{5a} , for example, denotes the partial augmentations of u with respect to the conjugacy class of elements of order 5). So

$\varepsilon_{1a} = \varepsilon_{2a} = 0$. Since all but one of the partial augmentations of $\pi(u)$, the image of u in $\mathbb{Z}S_5$, vanish, and a partial augmentation of $\pi(u)$ is the sum of the partial augmentations of u taken for classes which fuse in S_5 , we have

$$(1) \quad \begin{aligned} &\varepsilon_{4a}, \varepsilon_{4b}, \varepsilon_{8a} + \varepsilon_{8b}, \varepsilon_{3a} + \varepsilon_{6a}, \varepsilon_{5a} + \varepsilon_{10a}, \varepsilon_{12a} + \varepsilon_{12b} \in \{0, 1\}, \\ &|\varepsilon_{4a}| + |\varepsilon_{4b}| + |\varepsilon_{8a} + \varepsilon_{8b}| + |\varepsilon_{3a} + \varepsilon_{6a}| + |\varepsilon_{5a} + \varepsilon_{10a}| + |\varepsilon_{12a} + \varepsilon_{12b}| = 1. \end{aligned}$$

When u has order 2 or 4. Then the partial augmentations of u which are possibly nonzero are ε_{4a} , ε_{4b} , ε_{8a} and ε_{8b} (Remark 5). Thus $\chi_{11}(u) = \alpha(\varepsilon_{8a} - \varepsilon_{8b}) = -\sqrt{2}(\varepsilon_{8a} - \varepsilon_{8b})$. Also, $\chi_{11}(u)$ is the sum of fourth roots of unity. Since $\sqrt{2} \notin \mathbb{Q}(i)$ it follows that $\varepsilon_{8a} - \varepsilon_{8b} = 0$. Using $\varepsilon_{8a} + \varepsilon_{8b} \in \{0, 1\}$ from (1) we obtain $\varepsilon_{8a} = \varepsilon_{8b} = 0$. Now $|\varepsilon_{4a}| + |\varepsilon_{4b}| = 1$ by (1), so all but one of the partial augmentations of u vanish, with either $\varepsilon_{4a} = 1$ or $\varepsilon_{4b} = 1$. It follows that u is rationally conjugate to an element of G (necessarily of order 4).

When u has order 6 or 10. Then $u^3 = z$ or $u^5 = z$, respectively (Remark 4), i.e., zu is of order 3 or 5. Thus zu is, as already noted, rationally conjugate to an element of G , and hence the same holds for u itself.

When u has order 12. Then the partial augmentations of u which are possibly nonzero are ε_{4a} , ε_{4b} , ε_{8a} , ε_{8b} , ε_{3a} , ε_{6a} , ε_{12a} and ε_{12b} . The unit $\pi(u)$ has order 6 (Remark 5), so $\varepsilon_{12a} + \varepsilon_{12b} = 1$ and $\varepsilon_{4a} = \varepsilon_{4b} = \varepsilon_{8a} + \varepsilon_{8b} = \varepsilon_{3a} + \varepsilon_{6a} = 0$ by (1). Now $\chi_{11}(u) = -\sqrt{2}(\varepsilon_{8a} - \varepsilon_{8b})$ but $\sqrt{2} \notin \mathbb{Q}(\zeta_{12}) = \mathbb{Q}(i, \zeta_3)$, so $\varepsilon_{8a} = \varepsilon_{8b} = 0$ and consequently $\varepsilon_{8a} = \varepsilon_{8b} = 0$. Further $\chi_5(u) = 2(\varepsilon_{6a} - \varepsilon_{3a}) = 4\varepsilon_{6a} = \varepsilon_{6a}\chi_5(1)$, so if $\varepsilon_{6a} \neq 0$ then u is mapped under a representation of G affording χ_5 to the identity matrix or the negative of the identity matrix, leading to the contradiction $\chi_5(1) = \chi_5(u^6) = \chi_5(z) = -4$. Thus $\varepsilon_{3a} = \varepsilon_{6a} = 0$. So far, we have shown that ε_{12a} and ε_{12b} are the only possibly non-vanishing partial augmentations of u . We continue with a formal application of the Luthar–Passi method. Let ξ be a 12-th root of unity. Then

$$\mu(\xi, u, \chi_6) = \frac{1}{12}(\text{Tr}_{\mathbb{Q}(\zeta_{12})/\mathbb{Q}}(\chi_6(u)\xi^{-1}) + 6\mu(\xi^2, u^2, \chi_6) + \text{Tr}_{\mathbb{Q}(\zeta_{12}^3)/\mathbb{Q}}(\chi_6(u^3)\xi^{-3})).$$

Since u^3 is rationally conjugate to an element of order 4 in G , we have $\chi_6(u^3) = 0$. Since $\chi_6(u) = \beta(\varepsilon_{12a} - \varepsilon_{12b}) = (\zeta_{12}^7 - \zeta_{12}^{11})(\varepsilon_{12a} - \varepsilon_{12b})$, we have

$$\begin{aligned} \text{Tr}_{\mathbb{Q}(\zeta_{12})/\mathbb{Q}}(\chi_6(u)\zeta_{12}^{-7}) &= 6(\varepsilon_{12a} - \varepsilon_{12b}), \\ \text{Tr}_{\mathbb{Q}(\zeta_{12})/\mathbb{Q}}(\chi_6(u)\zeta_{12}^{-11}) &= -6(\varepsilon_{12a} - \varepsilon_{12b}). \end{aligned}$$

Next, $\chi_6(u^4) = 1$ since u^4 is rationally conjugate to an element of order 3 in G , and $\chi_6(u^6) = \chi_6(z) = -4$, from which it is easy to see that $\mu(\xi^2, u^2, \chi_6) = 1$ for a primitive 12-th root of unity ξ . Thus

$$\begin{aligned} \mu(\zeta_{12}^7, u, \chi_6) &= \frac{1}{2}((\varepsilon_{12a} - \varepsilon_{12b}) + 1) \geq 0, \\ \mu(\zeta_{12}^{11}, u, \chi_6) &= \frac{1}{2}(-(\varepsilon_{12a} - \varepsilon_{12b}) + 1) \geq 0, \end{aligned}$$

from which we obtain $|\varepsilon_{12a} - \varepsilon_{12b}| \leq 1$. Together with $\varepsilon_{12a} + \varepsilon_{12b} = 1$ it follows that $\varepsilon_{12a} = 0$ or $\varepsilon_{12b} = 0$. We have shown that all but one of the partial augmentations of u vanish.

When u has order 8. Then the partial augmentations of u which are possibly nonzero are ε_{4a} , ε_{4b} , ε_{8a} and ε_{8b} . Since $\pi(u)$ has order 4, its partial augmentations with respect to classes of elements of order 2 vanish and consequently $\varepsilon_{4a} = \varepsilon_{4b} = 0$. We have $\chi_{11}(u) = \alpha(\varepsilon_{8a} - \varepsilon_{8b}) = (-\zeta_8 + \zeta_8^3)(\varepsilon_{8a} - \varepsilon_{8b})$, and this time the Luthar–Passi method gives

$$\begin{aligned}\mu(\zeta_8^3, u, \chi_{11}) &= \frac{1}{2}((\varepsilon_{8a} - \varepsilon_{8b}) + 3) \geq 0, \\ \mu(\zeta_8, u, \chi_{11}) &= \frac{1}{2}(-(\varepsilon_{8a} - \varepsilon_{8b}) + 3) \geq 0,\end{aligned}$$

from which we obtain $|\varepsilon_{8a} - \varepsilon_{8b}| \leq 3$. Together with $\varepsilon_{8a} + \varepsilon_{8b} = 1$ it follows that $(\varepsilon_{8a}, \varepsilon_{8b}) \in \{(1, 0), (0, 1), (-1, 2), (2, -1)\}$. At this point we are stuck when limiting attention to complex characters only.

However, we may resort to p -modular characters. It is natural to choose $p = 5$ since \tilde{S}_5 is a subgroup of $\mathrm{SL}(2, 25)$. This can be seen as follows. The group $\mathrm{PSL}(2, 25)$ contains $\mathrm{PGL}(2, 5)$ as a subgroup (see [14, Kapitel II, 8.27 Hauptsatz]) which is isomorphic to S_5 and its pre-image in $\mathrm{SL}(2, 25)$ is isomorphic to \tilde{S}_5 (for example, since the Sylow 2-subgroups of $\mathrm{SL}(2, 25)$ are generalized quaternion groups). Let φ be the Brauer character afforded by a faithful representation $D : \tilde{S}_5 \rightarrow \mathrm{SL}(2, 25)$. The Brauer lift can be chosen such that $\varphi(x) = \alpha = -\zeta_8 + \zeta_8^3$ for an element x in the conjugacy class $8a$ of G (since $D(x)$ has determinant 1). Then x^5 lies in the class $8b$, and we obtain $\varphi(u) = \varepsilon_{8a}\varphi(x) + \varepsilon_{8b}\varphi(x^5) = (-\zeta_8 + \zeta_8^3)(\varepsilon_{8a} - \varepsilon_{8b})$. Since $\varphi(u)$ is the sum of two 8-th roots of unity, it follows that $|\varepsilon_{8a} - \varepsilon_{8b}| \leq 1$ and consequently $\varepsilon_{8a} = 0$ or $\varepsilon_{8b} = 0$. The proof is complete.

We remark that the choice of $p = 5$ is also strongly suggested by the general theory of cyclic blocks. The spin characters of \tilde{S}_5 form a single 5-block of defect 1, with Brauer graph (cf. [19, Theorem 4])

$$\begin{array}{ccccccccc} & \varphi_{4a} & & \varphi_{2a} & & \varphi_{2b} & & \varphi_{4b} & \\ \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ \\ \chi_6 & & \chi_{11} & & \chi_5 & & \chi_{12} & & \chi_7 \end{array}$$

Here, $\varphi = \varphi_{2a}$ and φ_{2b} is conjugate to φ under the Frobenius homomorphism.

3. THE GENERAL LINEAR GROUP $\mathrm{GL}(2, 5)$

We set $G = \mathrm{GL}(2, 5)$. Let z be a generator of $Z(G)$, which is a cyclic group of order 4. The quotient $G/\langle z \rangle$ is isomorphic to S_5 for which (ZC1) is known to hold. Let π denote the natural map $\mathbb{Z}G \rightarrow \mathbb{Z}G/\langle z \rangle$.

Let u be a nontrivial torsion unit in $V(\mathbb{Z}G)$. We will show that all but one of its partial augmentations vanish. For that, we use part of the character table of G , shown in Table 2 in the form obtained by requiring `CharacterTable("GL25")` in GAP [10], together with the natural 2-dimensional representation of G in characteristic 5. In Table 2, the row inscribed “in S_5 ” indicates to which classes in the quotient S_5 the listed classes of G are mapped. The classes omitted are the classes $2a$, $4a$, $4b$ of central 2-elements, and the classes $5a$, $20a$, $10a$, $20b$ of elements of order divisible by 5.

The characters χ_6 and χ_{16} have kernel $\langle z^2 \rangle$. The faithful characters χ_9 , χ_{14} , χ_{15} , χ_{21} and χ_{22} of G form a 5-block of G , with Brauer graph (cf. the theory of blocks of cyclic defect)

$$\begin{array}{ccccccccc} & \varphi_{4a} & & \varphi_{2a} & & \varphi_{2b} & & \varphi_{4b} & \\ \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ \\ \chi_{15} & & \chi_9 & & \chi_{21} & & \chi_{14} & & \chi_{22} \end{array}$$

class	1a	4c	2b	4d	4e	4f	4g	24a	12a	8a	6a	24b	3a	8b	24c	12b	24d
in S_5	1a	4a	2a	4a	4a	2a	4a	6a	3a	2b	3a	6a	3a	2b	6a	3a	6a
χ_2	1	i	-1	$-i$	$-i$	1	i	i	-1	$-i$	1	$-i$	1	i	i	-1	$-i$
χ_6	5	i	-1	$-i$	$-i$	1	i	$-i$	1	i	-1	i	-1	$-i$	$-i$	1	i
χ_{16}	4	$-i$	-1	$-2i$	1	i	1	$2i$	$-i$	-1	i
χ_9	6	α	.	$\bar{\alpha}$	$-\bar{\alpha}$.	$-\alpha$
χ_{14}	6	$-\alpha$.	$-\bar{\alpha}$	$\bar{\alpha}$.	α
χ_{15}	4	β	$-i$.	-1	$-\bar{\beta}$	1	.	$-\beta$	i	$\bar{\beta}$
χ_{21}	4	$2i$.	2	.	-2	.	.	$-2i$.
χ_{22}	4	$-\beta$	$-i$.	-1	$\bar{\beta}$	1	.	β	i	$-\bar{\beta}$

Irrational entries: $\alpha = 1 + i$,
 $\beta = -\zeta + \zeta^{17}$ where $\zeta = \exp(2\pi i/24)$.

TABLE 2. Part of the character table of $\text{GL}(2, 5)$

Set $\varphi = \varphi_{4a} = (\chi_{15} - \chi_9)|_{G_{5'}}$ (restriction to 5-regular elements). Then φ is an irreducible 5-modular Brauer character of G of degree 2 afforded by a natural representation $G \rightarrow \text{GL}(2, 5)$.

We remark that the remaining irreducible faithful characters of G form a 5-block of G which is algebraically conjugate to the one we consider.

We write $\varepsilon_{4c}, \varepsilon_{2b}, \dots, \varepsilon_{24d}$ for the partial augmentations of u at the classes listed in Table 2. We assume that u is not a central unit, so that its partial augmentations at central group elements are zero. It follows from Remark 4 that the order of u agrees with the order of some group element of G .

When the order of u is divisible by 5. Then $\pi(u)$ has order 5, by Remark 4, and u is the product of a unit of order 5 and a central group element of G . Since there is only one class of elements of order 5 in G , the 5-part of u is rationally conjugate to an element of G (Remark 5), and thus the same is valid for u .

When u has order 2. The group G has only one class of non-central elements of order 2, so Remark 5 applies.

When u has order 4. Then $\varepsilon_g(u) = 0$ for a group element g which is not a non-central element of order 2 or 4 (Remark 5). Evaluating the Brauer character φ at u gives

$$(2) \quad \varphi(u) = (\varepsilon_{4c} - \varepsilon_{4g})(1 + i) + (\varepsilon_{4d} - \varepsilon_{4e})(1 - i).$$

First, suppose that $\pi(u)$ has order 2. Then $u^2 = z^2$ (Remark 4), so $\varphi(u^2) = -2$ and $\varphi(u)$ is the sum of two primitive fourth roots of unity. These roots of unity are distinct since u is non-central in $\mathbb{Z}G$. Thus $\varphi(u) = i + (-i) = 0$ and (2) gives $\varepsilon_{4c} = \varepsilon_{4g}$ and $\varepsilon_{4d} = \varepsilon_{4e}$. Since (ZC1) holds for S_5 we have

$$\varepsilon_{4c} + \varepsilon_{4g} + \varepsilon_{4d} + \varepsilon_{4e} = 0, \quad \varepsilon_{2b} + \varepsilon_{4f} = 1.$$

From that we further obtain $\varepsilon_{4d} = -\varepsilon_{4c}$ and $\chi_2(u) = 1 - 2\varepsilon_{2b} + 4\varepsilon_{4c}i$. Since $|\chi_2(u)| = 1$ it follows that $\varepsilon_{4c} = 0$ and $\varepsilon_{2b} \in \{0, 1\}$. Thus all but one of the partial augmentations of u vanish.

Secondly, suppose that $\pi(u)$ has order 4. Then $\varphi(u^2) \neq -2$. Since $\varphi(u)$ is the sum of two distinct fourth roots of unity we have $|\varphi(u)| < 2$. Thus $\varphi(u) \in \{\pm(1+i), \pm(1-i)\}$ by (2). Since (ZC1) holds for S_5 we have

$$\varepsilon_{4c} + \varepsilon_{4g} + \varepsilon_{4d} + \varepsilon_{4e} = 1, \quad \varepsilon_{2b} + \varepsilon_{4f} = 0.$$

From that and (2) we further obtain that for some $a \in \mathbb{Z}$ and $\delta_i \in \{0, 1\}$, with exactly one δ_i nonzero, $\varepsilon_{4c} = a + \delta_1$, $\varepsilon_{4g} = a + \delta_2$, $\varepsilon_{4d} = a - \delta_3$ and $\varepsilon_{4e} = a - \delta_4$. Thus $\chi_2(u) = (\delta_1 + \delta_2 - \delta_3 - \delta_4)i - 2\varepsilon_{2b} + 4ai$ from which $\varepsilon_{2b} = 0$ and $a = 0$ follows. Thus all but one of the partial augmentations of u vanish.

When u has order 8. Then $\varepsilon_{8a} \neq 0$ or $\varepsilon_{8b} \neq 0$ by [7, Corollary 4.1] (an observation sometimes attributed to Zassenhaus). Since S_5 has no elements of order 8 we have $u^4 = z^2$ (by Remark 4).

Suppose that $\varepsilon_{8b} = -\varepsilon_{8a}$. Then $\chi_{16}(u) = -4\varepsilon_{8b}i$ (remember Remark 5). Since χ_{16} has degree 4 it follows that under a representation of G affording χ_{16} the unit u is mapped to a scalar multiple of the identity matrix. Thus the image of u in $\mathbb{Z}G/\langle z^2 \rangle$ is a central unit of order 4, the kernel of χ_{16} being $\langle z^2 \rangle$. But the center of $G/\langle z^2 \rangle$ has order 2, so we have reached a contradiction.

Hence $\varepsilon_{8a} + \varepsilon_{8b} \neq 0$, and since ε_{8a} and ε_{8b} are the classes of G which map onto class ε_{2b} in S_5 , in fact $\varepsilon_{8a} + \varepsilon_{8b} = 1$. Now $\chi_{16}(u) = 2(1 - 2\varepsilon_{8a})i$, and $|\chi_{16}(u)| \leq 4$ implies that $\varepsilon_{8a} \in \{0, 1\}$, so one of ε_{8a} and ε_{8b} vanish.

Next, we note that $\chi_9(u) = 0$: From $\chi_9(u^4) = \chi_9(z^2) = -\chi_9(1)$ we conclude that $\chi_9(u) \in \zeta_8\mathbb{Z}[i]$ for a primitive 8-th root of unity ζ_8 and a look at the character table shows that $\chi_9(u) \in \mathbb{Z}[i]$, but definitely $\zeta_8 \notin \mathbb{Z}[i]$.

Since

$$(3) \quad \chi_9(u) = (\varepsilon_{4c} - \varepsilon_{4g})(1+i) + (\varepsilon_{4d} - \varepsilon_{4e})(1-i)$$

and $\varepsilon_{4c} + \varepsilon_{4g} + \varepsilon_{4d} + \varepsilon_{4e} = 0$ it follows that $\varepsilon_{4c} = \varepsilon_{4g} = -\varepsilon_{4d} = -\varepsilon_{4e}$. Also $\varepsilon_{2b} + \varepsilon_{4f} = 0$. So $\chi_2(u) = (\pm 1 + 4\varepsilon_{4c})i - 2\varepsilon_{2b}$ which implies $\varepsilon_{2b} = 0$ and $\varepsilon_{4c} = 0$, and we are done.

When u has order 3. The group G has only one class of elements of order 2, so Remark 5 applies.

When u has order 6. The only partial augmentations of u which are possibly nonzero are ε_{2b} , ε_{3a} and ε_{6a} . Since the class ε_{6a} maps in S_5 to the class of elements of order 3 it follows that $\pi(u)$ is rationally conjugate to a group element of order 3 in S_5 . Hence u is the product of z^2 and a unit of order 3 (Remark 4), and u is rationally conjugate to a group element.

When u has order 12. Only partial augmentations of u taken at classes of elements of order 2, 4, 3, 6 and 12 are possibly nonzero. The classes of elements of order 3, 6 and 12 map in S_5 to the class of elements of order 3. Thus $\pi(u)$ is of order 3 and u is the product of z and a unit of order 3, so u is rationally conjugate to a group element.

When u has order 24. Then $\pi(u)$ is rationally conjugate to an element of order 6 in S_5 , and so

$$\begin{aligned}
 &\varepsilon_{24a} + \varepsilon_{24b} + \varepsilon_{24c} + \varepsilon_{24d} = 1, \\
 &\varepsilon_{12a} + \varepsilon_{6a} + \varepsilon_{3a} + \varepsilon_{12b} = 0, \\
 (4) \quad &\varepsilon_{8a} + \varepsilon_{8b} = 0, \\
 &\varepsilon_{4c} + \varepsilon_{4d} + \varepsilon_{4e} + \varepsilon_{4g} = 0, \\
 &\varepsilon_{2b} + \varepsilon_{4f} = 0.
 \end{aligned}$$

From $\chi_9(u^{12}) = -\chi_9(1)$ we conclude that $\chi_9(u) \in \zeta_8 \mathbb{Z}[i, \zeta_3]$ for a primitive 8-th root of unity ζ_8 and a primitive third root of unity ζ_3 . A look at the character table shows $\chi_9(u) \in \mathbb{Z}[i]$, so $\chi_9(u) = 0$ as $\zeta_8 \notin \mathbb{Z}[i, \zeta_3]$. In the same way we argue that $\chi_{21}(u) = 0$.

Thus evaluation (3) of $\chi_9(u)$ is zero, and with (4) it follows that $\varepsilon_{4c} = \varepsilon_{4g} = -\varepsilon_{4d} = -\varepsilon_{4e}$. Now $(\chi_2 + \chi_6)(u) = -4\varepsilon_{2a} + 8\varepsilon_{4c}i$. Since $\chi_2 + \chi_6$ has degree 6 we conclude that $\varepsilon_{4c} = 0$.

We have $0 = \chi_{21}(u) = 2(\varepsilon_{6a} - \varepsilon_{3a}) + 2i(\varepsilon_{12a} - \varepsilon_{12b})$, so $\varepsilon_{6a} = \varepsilon_{3a}$ and $\varepsilon_{12a} = \varepsilon_{12b}$. Further $\varepsilon_{6a} = -\varepsilon_{12a}$ from (4). Thus $\chi_{16}(u) \in -4\varepsilon_{12a} + i\mathbb{Z}$. From $\chi_{16}(u^6) = -\chi_{16}(1)$ we obtain $\chi_{16}(u) \in i\mathbb{Z}[\zeta_3]$. It follows that $-4\varepsilon_{12a}i \in \mathbb{Z}[\zeta_3]$ and $\varepsilon_{12a} = 0$. Now $\chi_2(u) \in -2\varepsilon_{2b} + i\mathbb{Z}$ and so $\varepsilon_{2b} = 0$. Also $(\chi_2 + \chi_{16})(u) = -2\varepsilon_{2b} - 6\varepsilon_{8a}i$ and since $\chi_2 + \chi_{16}$ has degree 5 we have $\varepsilon_{8a} = 0$.

Set $a = \varepsilon_{24a} + \varepsilon_{24c}$ and $b = \varepsilon_{24b} + \varepsilon_{24d}$. Then $\chi_2(u) = (a - b)i$ and thus $a - b = \pm 1$. Together with $a + b = 1$ it follows that $(a, b) = (1, 0)$ or $(a, b) = (0, 1)$. In the first case, $\chi_{15}(u) = (2\varepsilon_{24a} - 1)\beta - 2\varepsilon_{24b}\beta$, and in the second $\chi_{15}(u) = 2\varepsilon_{24a}\beta + (1 - 2\varepsilon_{24b})\beta$. Using the sum formula for sin with $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$ it is easiest to calculate $\beta = -\sqrt{\frac{3}{2}}(1 + i)$. In particular, $|\beta| = \sqrt{3}$. Since $\chi_{15}(u)$ is the sum of four roots of unity, it is readily seen that if $\chi_{15}(u)$ assumes the first value, then $\varepsilon_{24b} = 0$ and $\varepsilon_{24a} \in \{0, 1\}$, and if $\chi_{15}(u)$ assumes the second value, then $\varepsilon_{24a} = 0$ and $\varepsilon_{24b} \in \{0, 1\}$. It follows that exactly one of ε_{24a} , ε_{24c} , ε_{24b} and ε_{24d} is nonzero, and we are done. We remark that the last argument can be replaced by a simpler “modular” argument: we already know that $\chi_{15}(u)$ agrees with the value of χ_{15} at a class of elements of order 24 since $\chi_9(u) = 0$ and $\varphi = \chi_9 - \chi_{15}$ on 5-regular elements.

REFERENCES

- [1] S. D. Berman, *On the equation $x^m = 1$ in an integral group ring*, Ukrain. Mat. Ž. **7** (1955), 253–261.
- [2] C. Bessenrodt, *Representations of the covering groups of the symmetric groups and their combinatorics*, Sémin. Lothar. Combin. **33** (1994), Art. B33a, approx. 29 pp. (electronic).
- [3] V. Bovdi, C. Höfert, and W. Kimmerle, *On the first Zassenhaus conjecture for integral group rings*, Publ. Math. Debrecen **65** (2004), no. 3-4, 291–303.
- [4] V. A. Bovdi, E. Jespers, and A. B. Konovalov, *Integral group ring of the first Janko simple group*, ArXiv eprint ([\protect\vrule width0pt\protect\href{http://arxiv.org/abs/math.GR/0608441}](http://arxiv.org/abs/math.GR/0608441)){<http://arxiv.org/abs/math.GR/0608441>} version 2, 18 Aug 2006.
- [5] V. A. Bovdi and A. B. Konovalov, *Integral group ring of the first Mathieu simple group*, ArXiv eprint ([\protect\vrule width0pt\protect\href{http://arxiv.org/abs/math.RA/0605316}](http://arxiv.org/abs/math.RA/0605316)){<http://arxiv.org/abs/math.RA/0605316>} version 1, 12 May 2006.

- [6] V. Bovdi, A. Kononov, R. Rossmanith, C. Schneider, *LAGUNA – Lie Algebras and UNits of group Algebras*, v.3.3.3; 2006 ([\protect\href{http://ukrgap.exponenta.ru/laguna.htm}](http://ukrgap.exponenta.ru/laguna.htm)){<http://ukrgap.exponenta.ru/laguna>.
- [7] J.A. Cohn and D. Livingstone, *On the structure of group algebras. I*, Canad. J. Math. **17** (1965), 583–593. MR MR0179266 (31 #3514)
- [8] M.A. Dokuchaev and S.O. Juriaans, *Finite subgroups in integral group rings*, Canad. J. Math. **48** (1996), no. 6, 1170–1179.
- [9] M.A. Dokuchaev, S.O. Juriaans, and C. Polcino Milies, *Integral group rings of Frobenius groups and the conjectures of H. J. Zassenhaus*, Comm. Algebra **25** (1997), no. 7, 2311–2325.
- [10] The GAP Group, *GAP — Groups, Algorithms, and Programming, Version 4.4*, 2005, ([\protect\href{http://www.gap-system.org}](http://www.gap-system.org)){<http://www.gap-system.org>}).
- [11] M. Hertweck, *Partial augmentations and Brauer character values of torsion units in group rings*, submitted, 2005.
- [12] ———, *On the torsion units of some integral group rings*, Algebra Colloq. **13** (2006), no. 2, 329–348.
- [13] G. Higman, *Units in group rings*, Ph.D. thesis, University of Oxford, Balliol College, 1940.
- [14] B. Huppert, *Endliche Gruppen. I*, Die Grundlehren der Mathematischen Wissenschaften, Band 134, Springer-Verlag, Berlin, 1967.
- [15] S.O. Juriaans and C. Polcino Milies, *Units of integral group rings of Frobenius groups*, J. Group Theory **3** (2000), no. 3, 277–284.
- [16] I. S. Luthar and I. B. S. Passi, *Zassenhaus conjecture for A_5* , Proc. Indian Acad. Sci. Math. Sci. **99** (1989), no. 1, 1–5.
- [17] I. S. Luthar and Poonam Trama, *Zassenhaus conjecture for S_5* , Comm. Algebra **19** (1991), no. 8, 2353–2362.
- [18] Z. Marciniak, J. Ritter, S. K. Sehgal, and A. Weiss, *Torsion units in integral group rings of some metabelian groups. II*, J. Number Theory **25** (1987), no. 3, 340–352.
- [19] A. O. Morris and A. K. Yaseen, *Decomposition matrices for spin characters of symmetric groups*, Proc. Roy. Soc. Edinburgh Sect. A **108** (1988), no. 1-2, 145–164.
- [20] D. Passman, *Permutation groups*, W. A. Benjamin, Inc., New York-Amsterdam, 1968.
- [21] I. Schur, *Über die Darstellung der symmetrischen und alternierenden Gruppe durch gebrochen lineare Substitutionen*, J. Reine Angew. Math. **139** (1911), 155–250.

INSTITUTE OF MATHEMATICS, UNIVERSITY OF DEBRECEN, 4010 DEBRECEN, P.O. BOX 12,
 HUNGARY; INSTITUTE OF MATHEMATICS AND INFORMATICS, COLLEGE OF NYÍREGYHÁZA, SÓSTÓI
 ÚT 31/B, 4410 NYÍREGYHÁZA, HUNGARY
E-mail address: vbovdi@math.klte.hu

UNIVERSITÄT STUTTGART, FACHBEREICH MATHEMATIK, IGT, PFAFFENWALDRING 57, 70550
 STUTTGART, GERMANY
E-mail address: hertweck@mathematik.uni-stuttgart.de